# GENERALIZED NORMAL WINDOW FOR DIGITAL SIGNAL PROCESSING

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## ABSTRACT

The bandwidth and smoothness of windows play an important role in digital signal processing. In applications such as manufacturing process and quality monitoring, to radar target tracking and cellular communications, the design of appropriate windows is one of the crucial steps. In this paper the generalized normal function is introduced as a smooth and configurable window. With the aid of illustrations, the advantage of using this window over conventional windows is discussed.

## 1. INTRODUCTION

In concept, any mathematical representation of a signal is infinite. Truncating any portion of the infinite signal alters its frequency content. Thus, the frequency content of the truncating function plays an important role in the final spectral shape of the truncated signal. In signal processing, the truncating function is referred to as a window, and the operation of truncating is referred to as windowing. Currently, there are a variety of known windows and are often chosen based on their spectral behavior. The choice of window affects the detectability and resolvability of specific spectral components, dynamic range, confidence, ease of implementation and may also govern spectral leakage [1]. A comprehensive table of windows along with their figures of merit is provided in page 176 of [1, 2].

Windows have tunable parameters, of which the most common ones include length and time-location of the window. For example, in a rectangular window $U(t-\tau) - U(t-\tau-T)$ (where $U(t)$ is a unit step function), $\tau$ is the time delay and $T$ is the duration. In a Gaussian window $\exp\left(-\frac{(t-\tau)}{s}^2\right)$, $\tau$ is the time delay and $s^2$ is the time scale. The choice of windows are also based on the mainlobe width and the sidelobes levels [3] (see Figure 1).

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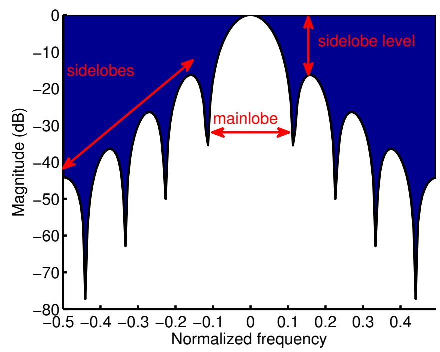



**Fig. 1**. Discrete Fourier transform of a Gaussian window showing mainlobe and sidelobes.

The Gaussian window is the most concentrated, simultaneously in time and frequency, and attains the uncertainty lower bound [4]. However, it has the widest mainlobe and offers the least spectral resolution of nearby kernels [1]. On the other hand, the rectangular window provides the best spectral resolution of nearby kernels (since it has the narrowest mainlobe) but are unusable in restricted bandwidth applications because of high sidelobe levels and spectral leakage. Several other windows like the triangle window, trapezoidal window, Hann window and Hamming window offer a compromise between the spectral resolution and spectral leakage but, they do not have a constant time amplitude like the rectangular window. These windows attenuate the frequency components appearing at the ends of signals with time varying spectrum. Many of these windows are defined such that they are not differentiable at all time positions. In this paper, we are proposing the use of reparameterized Generalized normal (GN) function as a window. Recently, the GN function has attracted attention in the mathematics and the statistics community as a smooth function and a probability density function [5–8]. The GN window is continuous and differentiable, and with judicious choice of parameters, one can control the bandwidth, time-location, duration and *flatness* of the window. This would be a preferred window to use when analyzing signals with time varying spectral content, since the amplitude attenuation at any given time is parameterized and is customizable based on the application. It provides a set of windows between the two extremes: a Gaussian and a rectangular window.

The remainder of this paper is organized as follows. First, we provide the time domain mathematical form and discuss it characteristics. Next, we discuss the frequency domain representation in a numerically stable logarithmic form. Illustrations have been provided for visual representation of different realizations of the GN window family. Finally, a note on application is provided with time-frequency representation.

## 2. TIME DOMAIN REPRESENTATION

### 2.1. The Generalized Normal Function

The GN function (GNF) is a smooth, even function with infinite derivatives. It is defined as

$$\text{GNF}\,(t;\tau,s,n) \triangleq \exp\left(-\left(\frac{t-\tau}{s}\right)^n\right), \qquad (1)$$

where $t \in \mathbb{R}$ is the domain, $\tau \in \mathbb{R}$ is the shift in the point of even symmetry, $s \in \mathbb{R}^{++}$ is the spread of the function and $n \in \{2\eta; \forall\, \eta \in \mathbb{N}\}$ is the order of the function[1]. The range of this function is between 0 and 1, $\text{GNF}\,(t) \in (0,1]\,\forall t$. The supremum of this function $\sup(\text{GNF}\,(t;s,n,\tau)) = 1$ is at $t=\tau$, and the infimum $\inf(\text{GNF}\,(t;s,n,\tau)) = 0$ exists at $t=\pm\infty$, for all valid $n$ and $s$. The function exists for all real $n$ if the absolute value of $t-\tau$ is used but, the choice of even $n$ ensures smoothness.

The widely used Gaussian function is the order $n = 2$ GNF. In the limit $n \to \infty$, the GNF approaches a rectangular function, i.e.

$$\lim_{n\to\infty} \text{GNF}\,(t;\tau,s,n) = \begin{cases} 1 & |t-\tau| < s \\ e^{-1} & |t-\tau| = s \\ 0 & |t-\tau| > s \end{cases} \qquad (2)$$

For intermediate values of $n = 4, 6, 8, \ldots$, the function takes the shape as shown in Figure 2.

### 2.2. The Generalized Normal window

Since the GNF is smooth and has adjustable sidelobes levels, it is a suitable choice for a signal window. The window derived by reparameterizing GNF will be referred to as Generalized normal window (GNW).

Designing a window requires parametric representation that bears temporal significance. Using $s$ and $n$ are not suitable window parameters since they do not allow direct specification of the start and the stop times of the window. Since

[1] $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^{++}$ is the set of strictly positive real numbers and $\mathbb{N}$ is the set of natural numbers

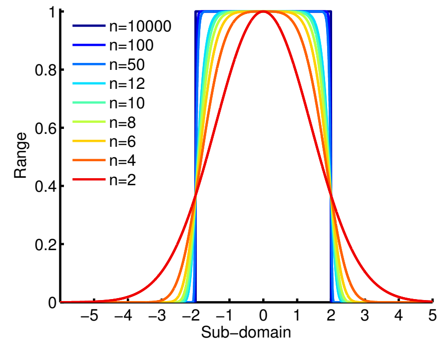



**Fig. 2**. Generalized Normal Function for different values of $n$, with $s = 2$ and $\tau = 0$.

the window attains 1 at $t=\tau$, and 0 at $t=\pm\infty$, we need to define a tolerance $\varepsilon$, such that $1-\varepsilon \approx 1$ and $\varepsilon \approx 0$. If we consider $t_o$ as the time of origin, $t_s$ as the time of saturation, and $t_e$ as the end time of the window, we can define the GN window with these parameters $\text{GNW}\,(t;t_o,t_s,t_e,\varepsilon)$. Then, the following assumptions hold:

$$\text{GNW}\,(t_o) = \varepsilon \qquad (3a)$$
$$\text{GNW}\,(t_s) = 1-\varepsilon \qquad (3b)$$
$$\text{GNW}\,(t_e) = \text{GNW}\,(t_o)\,. \qquad (3c)$$

The time locations $t_o, t_s, t_e$ and $\tau$ are labeled in Figure 3.

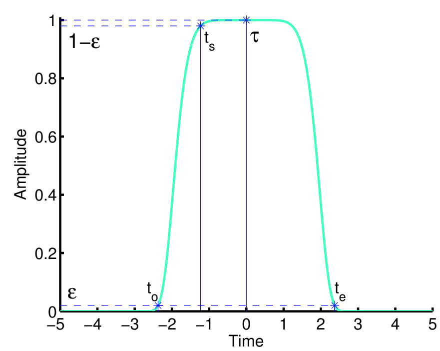



**Fig. 3**. The GN window with exaggerated $\varepsilon$ for clarity.

Using the equations in (1) and (3), we can form the following system of equations:

$$\varepsilon = \exp\left(-\left(\frac{t_o-\tau}{s}\right)^n\right) \qquad (4a)$$
$$\varepsilon = \exp\left(-\left(\frac{t_e-\tau}{s}\right)^n\right) \qquad (4b)$$
$$1-\varepsilon = \exp\left(-\left(\frac{t_s-\tau}{s}\right)^n\right). \qquad (4c)$$

The corresponding GN function parameters $s$, $n$ and $\tau$ can be computed as

$$\tau = \frac{t_o + t_e}{2} \quad (5a)$$

$$n = \frac{\log\left(\frac{\log(1-\hat{\varepsilon})}{\log(\hat{\varepsilon})}\right)}{\log\left(\frac{t_s-\tau}{t_o-\tau}\right)} \quad (5b)$$

$$s = \frac{-t_o + \tau}{\sqrt[n]{-\log(\hat{\varepsilon})}} \quad (5c)$$

with a choice of the largest $\hat{\varepsilon} \leq \varepsilon$ such that $n$ is an even integer. Note that, (5b) is a strictly decreasing function of $\varepsilon$ with a strictly increasing slope. Also, for large $n$, rounding its value to the nearest even number barely changes the shape of the window. The user may choose this approach to simplify the implementation. Figure 4 shows an example GNW that starts at $t_o = 2$s, saturates in another 0.5s at $t_s = 2.5$s and ends at $t_e = 8$s with an $\varepsilon \approx 0.01$.

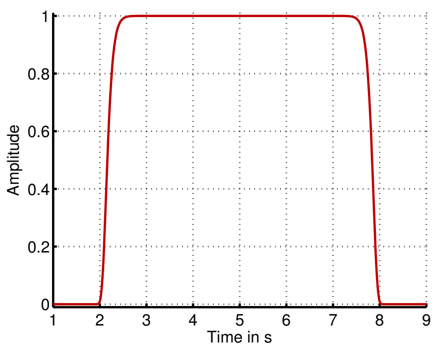



**Fig. 4**. Time domain plot of GNW $(t; 2, 2.5, 8, 0.01)$.

## 3. FREQUENCY DOMAIN REPRESENTATION

The GNF does not have a generalized closed form representation of the Fourier transform. In [7, 8], the authors have derived an analytic Fourier transform for the GNF. Specifically, the Fourier transform of the GNF has been derived in [7] as

$$\overline{\text{GN}}\left(\omega; 0, 1, n\right) = \frac{2}{n}\sum_{k=0}^{\infty}(-1)^k\,\Gamma\left(\frac{2k+1}{n}\right)\frac{\omega^{2k}}{(2k)!}, \quad (6)$$

where $\omega$ is frequency in rad/s. Using Fourier transform properties, the appropriate time shift and scaling can be introduced[2],

$$\overline{\text{GN}}\left(\omega; \tau, s, n\right) = s\, e^{-j\omega\tau}\,\overline{\text{GN}}\left(\omega\, s; 0, 1, n\right) \quad (7)$$

For larger values of $\omega$, Equation (7) may become numerically unstable. The numerically preferred approach is to compute the frequency in log scale

$$\overline{\text{GN}}\left(\omega; \tau, s, n\right) = \begin{cases} \sum_{k=0}^{\infty}(-1)^k \exp\Big[\log\left(\frac{2s}{n}\right) + \log\Gamma\left(\frac{2k+1}{n}\right) + 2k\log(ws) - \log\Gamma\left(2k+1\right) - jw\tau\Big] & ; w \neq 0 \\ \frac{2s}{n}\,\Gamma\left(\frac{1}{n}\right) & ; \lim\omega \to 0 \end{cases} \quad (8)$$

In fact, using Equation (8) was observed to be about 2.7 times faster when implemented in MATLAB 2012a on a Core i7 Windows laptop. To accurately compute the frequency components at larger $f$, an extended precision numerical solver is needed. The authors in [7] discussed the convergence of this series.

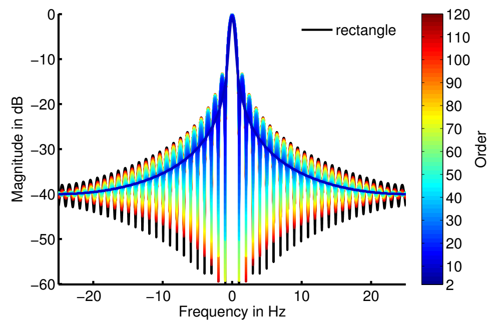



**Fig. 5**. Discrete Fourier transforms of GNF for different values of n, with $s = 1$.

## 4. DISCUSSION ON APPLICATIONS

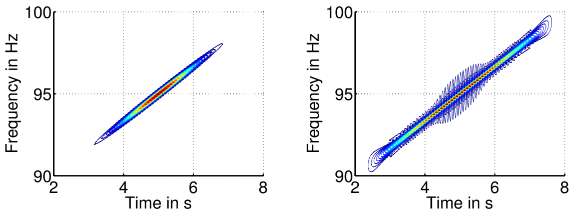
The GNW is the best choice in applications where a smooth truncation signals in time domain, with time-varying spectrum, is required. Consider the Wigner Ville time-frequency representation (TFR) [9] of a Gaussian windowed linear chirp as shown in Figure 6(a). The Gaussian window not only attenuates the time amplitude at the end points, but also alters the spectrum by attenuating the frequency components existing at those times. Similar effects (though not as severe) would be observed with any other window that does not have a constant amplitude time response like the rectangular window. On the other hand, the GNW can be designed to custom fit the acceptable level of frequency attenuation to achieve the desired level of smoothness. Figure 6(b) shows the Wigner Ville distribution of a GN windowed chirp. Note that the GNW was able to deliver better frequency resolution than the Gaussian window without generating a dog-bone effect [9] at the end points.

[2] Here $j$ is the complex representation of $\sqrt{-1}$

limited window. The GNW would provide an excellent alternative that is smooth and configurable.

In manufacturing industries, automatized process monitoring and quality inspection is performed by real time monitoring of sensor signals. The GNW is an excellent fit as a smooth window that allows for minute control on signal distortion.

Windows are also used as interpolating functions. For instance, the appropriate choice of window was critical in improving the effective frequency measurement resolution in [13] by controlling the spectral leakage. The GNW would be a well suited option here.

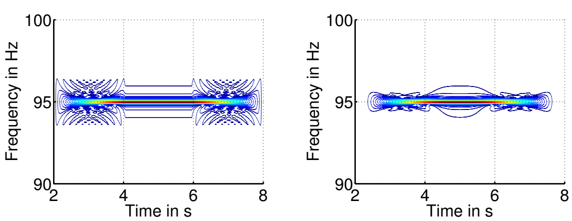



(a) Gaussian windowed chirp. (b) GNW $(t; 2, 3, 8, 0.01)$ windowed chirp.

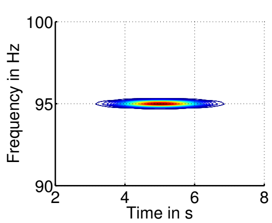
**Fig. 6**. Wigner Ville time-frequency representation of a linear chirp spanning 90-100Hz in 2-8s with a window spanning from 2s to 8s.

When constructing a time-frequency based matching pursuit decomposition dictionary [4, 10] some applications require a chirp dictionary and the GN window could be used to custom fit the dictionary to an application. Even when constructing a dictionary of pure sinusoids, using a GNW as opposed to the traditional Gabor atoms would add flexibility to dictionary design with explicit control on the bandwidth.

When a signal has time-invariant frequency components, like the sine wave shown in Figure 7, the rectangular window is the best choice since it provides the best frequency resolution. Figures 7(a),(b) and (c) show rectangular windowed sinusoid, GN windowed sinusoid, and Gaussian windowed sinusoid. Note the cross terms due to the dog-bone effect [9] in the rectangular windowed sinusoid, which can be attenuated by using the appropriate choice of the GN window.




(a) Rectangular windowed sinusoid. (b) GNW $(t; 2, 3, 8, 0.01)$ windowed sinusoid.




(c) Gaussian windowed sinusoid.

**Fig. 7**. Wigner Ville time-frequency representation of a 95 Hz sinusoid with a window spanning from 2s to 8s.

In waveform design for radars, authors have used a trapezoidal window [11, 12] and on occasion a sigmoid function envelop [12] to replace rectangular window with a more band

## 5. CONCLUSION

A function family, generalized normal function, has been adopted for use as a signal window. The function has been reparameterized such that the window shape could be accurately controlled based on specific time parameters and accuracy. This window, referred to as the generalized normal window provides a selectable tradeoff between spectral resolution and band limitedness and allows all the intermediate choice between the two extremes: the Gaussian window and the rectangular window.

While implementing this window in time or frequency domain is possible when represented in sampled time, the analytic computation in frequency domain remains a numerical challenge. Double precision number representation is insufficient to compute the Fourier transform analytically in the high frequency range. Subsequently, these numerical challenges can be extended to computation of time-frequency representations. For example, the plots of frequency and time-frequency representations generated in this paper were computed numerically from a sampled time instantiation of the window. While the authors verified the accuracy of presented analytic frequency representation[3], it could not be used to accurately generate a plot in MATLAB. A numerically stable approxiamtion method to compute the Fourier transform would surely enhance the usability of this window.

[3]The accuracy was verified using extended precision in Mathematica 8.0

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